Factorial Experiments

The 2 x 2 Factorial

Factorial experiment with 2 factors A and B, which each have 2 levels: a_1, a_2 and b_1, b_2

There are 4 (2 x 2) treatments:

1. $a_1\ b_1$
2. $a_1\ b_2$
3. $a_2\ b_1$
4. $a_2\ b_2$

Factorial Experiments

Conceptual Layout

	B_j			
	1		2	
A_i	1	2	1	2
R_k	1	…	…	…
	…	…	…	…
	K	…	…	IJK

Factorial Experiments

Linear Additive Model

$$Y_{ijk} = \mu + A_i + B_j + AB_{ij} + \varepsilon_{(ij)k}$$

Where:
- Y_{ijk} = variable to be analyzed from the k^{th} experimental unit
- μ = overall mean
- A_i = effect of the i^{th} level of A
- B_j = effect of the j^{th} level of B
- AB_{ij} = interaction effect of the effect of the i^{th} level of A with effect of the j^{th} level of B
- $\varepsilon_{(ij)k}$ = experimental error associated with k^{th} experimental unit, NID(0,σ^2)

Factorial Experiments

ANOVA

Degrees of Freedom:

$A_i = (I - 1)$

$B_j = (J - 1)$

$AB_{ij} = (I - 1)(J - 1)$

$\varepsilon_{(ij)k} = IJ(K - 1)$

Total = IJK - 1

Factorial Experiments
ANOVA

Sums of Squares Algorithm:

1. write the formula for the df
2. expand the product
   - each combination of subscripts corresponds to a mean
   - the sign preceding the mean is retained
   - replace all missing subscripts for each mean with dots (.)
   - a value of 1 in the expansion corresponds to the overall mean
3. SS are equal to the linear combination of means summed over all of the data
4. subscripts not appearing in the linear combination are replaced by the corresponding capital letter

Factorial Experiments
ANOVA

Factor A SS:

(I - 1)

$(\overline{y}_{i}.. - \overline{y}...)^2$

$$SS(A) = JK\sum_{i=1}^{I}(\overline{y}_{i}.. - \overline{y}...)^2$$

Factorial Experiments

ANOVA

Factor B SS:

(J - 1)

$(\overline{y}_{.j.} - \overline{y}_{...})^2$

$$SS(B) = IK\sum_{j=1}^{J}(\overline{y}_{.j.} - \overline{y}_{...})^2$$

Factorial Experiments

ANOVA

AB Interaction SS:

(I - 1)(J - 1)

IJ - I - J + 1

$(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...})^2$

$$SS(AB) = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...})^2$$

$$SS(AB) = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{y}_{ij.} - \overline{y}_{...})^2 - SS(A) - SS(B)$$

Factorial Experiments
ANOVA

Error SS:

IJ(K - 1)

IJK - IJ

$$SS(error) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \overline{y}_{ij}.)^2$$

Factorial Experiments
ANOVA

Total SS:

IJK - 1

$$SS(total) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \overline{y}...)^2$$

Factorial Experiments
ANOVA

Mean Squares:

MS(A) = SS(A)/(I - 1)

MS(B) = SS(B)/(J -1)

MS(AB) = SS(AB)/[(I - 1)(J - 1)]

MS(error) = SS(error)/[IJ(K - 1)]

Factorial Experiments
ANOVA

F-Tests:

F(A) = MS(A)/MS(error)

F(B) = MS(B)/MS(error)

F(AB) = MS(AB)/MS(error)

* Assuming both A and B are fixed factors

Factorial Experiments

ANOVA

Source	df	SS	MS
A_i	I - 1	$JK\sum_{i=1}^{I}(\bar{y}_{i..}-\bar{y}_{...})^2$	SS(A)/(I - 1)
B_j	J - 1	$IK\sum_{j=1}^{J}(\bar{y}_{.j.}-\bar{y}_{...})^2$	SS(B)/(J - 1)
AB_{ij}	(I - 1)(J - 1)	$K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	SS(AB)/(I - 1)(J -1)
$\varepsilon_{(ij)k}$	IJ(K – 1)	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\bar{y}_{ij.})^2$	SS(ε)/ IJ(K - 1)
Total	IJK - 1	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-\bar{y}_{...})^2$	

Factorial Experiments

Example Calculations

	Nitrogen (j)		Means (i)
Hybrid (i)	75	150	
A	177.1	166.6	174.9
	160.2	189.6	
	159.7	196.0	
Means (ij)	165.7	184.1	
B	181.6	188.9	183.6
	169.3	207.8	
	167.1	186.9	
Means (ij)	172.7	194.5	
Means (j)	169.2	189.3	179.2

Factorial Experiments
Example Calculations

$$SS(total) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}...)^2$$

$$= (177.1 - 179.2)^2 + \ldots + (186.9 - 179.2)^2$$

$$= 2516.3267$$

$$SS(error) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij}.)^2$$

$$= (177.1 - 165.7)^2 + \ldots + (186.9 - 194.5)^2$$

$$= 1062.4467$$

Factorial Experiments
Example Calculations

$$SS(H) = JK\sum_{i=1}^{I}(\bar{y}_{i}.. - \bar{y}...)^2$$

$$= (2)(3)[(174.9 - 179.2)^2 + (183.6 - 179.2)^2]$$

$$= 228.8133$$

$$SS(N) = IK\sum_{j=1}^{J}(\bar{y}_{.j}. - \bar{y}...)^2$$

$$= (2)(3)[(169.2 - 179.2)^2 + (189.3 - 179.2)^2]$$

$$= 1216.0533$$

Factorial Experiments
Example Calculations

$$SS(trt) = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij}. - \bar{y}...)^2$$

$$= (3)[(165.7 - 179.2)^2 + \ldots + (194.5 - 179.2)^2]$$

$$= 1453.8800$$

$$SS(HN) = SS(Trt) - SS(H) - SS(N)$$

$$= 1453.8800\text{-}228.8133\text{-}1216.0533$$

$$= 9.0133$$

Factorial Experiments
Example Calculations

Source	df	SS	MS	F	P > F
Hybrid	1	228.8133	228.8133	1.72	0.226
Nitrogen	1	1216.0533	1216.0533	9.16	0.016
H x N	1	9.0133	9.0133	0.07	0.801
Error	8	1062.4467	132.8058		
Total	11	2516.3267			

Factorial Experiments

Contrasts

Source	Treatments			
	H_1N_1	H_1N_2	H_2N_1	H_2N_2
Hybrid	1	1	-1	-1
Nitrogen	1	-1	1	-1
H x N	1	-1	-1	1

Factorial Experiments

Contrasts

$$SS(Q) = \frac{r(\sum c_i \bar{Y}_i)^2}{\sum c_i^2}$$

$$SS(H) = \frac{3(165.7 + 184.1 - 172.7 - 194.5)^2}{4} = 228.8133$$

$$SS(N) = \frac{3(165.7 - 184.1 + 172.7 - 194.5)^2}{4} = 1216.0533$$

$$SS(HN) = \frac{3(165.7 - 184.1 - 172.7 + 194.5)^2}{4} = 9.0133$$